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Interval Graphs and its Applications

Abstract

Extensive study and research has been done on interval graphs for several decades by both mathematicians and computer scientists. These graphs are used to provide numerous models in diverse areas such as genetics, psychology, sociology, archaeology, engineering, transportation and scheduling. A graph is an interval graph if it is the intersection graph of intervals on a line. Interval graphs are known to be intersection of chordal graphs and as teroidal triple –free graph. This paper studies interval graph , its characterization and consecutive 1's property of the clique matrix. Gilmore and Hoffman have given the relationship between the interval graph , comparability graph and its clique matrix. It also gives characterization of interval graph through Lekkerkerker's theorem. It gives a complexity analysis of consecutive 1's testing. It discuss applications of interval graph for chemical compounds which must be refrigerated under closed monitored conditions and also elaborate another application of storing of records through consecutive retrieval property.

M.G.Mohanan
Associate Professor,
Deptt. of Mathematics
and Computers,
Vivekcollege,Goregoan (W),
University of Mumbai

Latha Mohanan
Assistant Professor,
Deptt. of Mathematics and
Statistics,
Guru Nanak Khalsa College,
University of Mumbai

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Introduction

An undirected graph G is called an Interval Graph if its vertices can be put into one-to-one correspondence with a set of intervals $\mathcal{J}$of linearly ordered sets such that two vertices are connected by an edge of G if and only if their corresponding intervals have nonempty intersection. We call $\mathcal{J}$an interval representation for G. It is not important whether we use open intervals or closed intervals, the resulting class of graphs will be the same. Figure 1 shows an interval graph - the windmill graph and an interval representation for it.

Triangulated Graph Property

Every simple cycle of length strictly greater than 3 possesses a chord. Graphs which satisfy this property are called triangulated graph. The graph in Figure1is triangulated, but the house graph in Figure 2 is not triangulated because it contains a chordless 4-cycle.

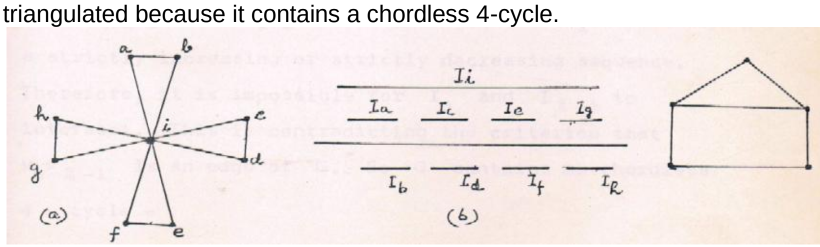

Figure 1 Figure 2

Some characterization of interval graphs

The following theorem and its corollary will establish where the class of interval graphs belongs in the world of perfect graphs.

Theorem 1.(Gilmore Hoffman).[3],[4]

Let G be an undirected graph. The following statements are equivalent.

(i) G is an interval graph
(ii) G contains no chordless 4-cycle and its complement $\bar{G}$is comparability graph.
(iii) The maximal cliques of G can be linearly ordered such that, for every vertex x of G, the maximal cliques containing x occur consecutively.

Proof (i) $\Longrightarrow$ (ii)

Suppose the interval graph G contains a chordless cycle [v0,v1,v2,….vℓ-1,v0] with ℓ> 3. Let Ik denote the interval corresponding to vk for i = 1,2,…. ℓ - 1, choose a point P1 ϵ Ii-1∩ Ii . Since Ii-1 and Ii+1do not overlap, the P1 constitute a strictly increasing or strictly decreasing sequence. Therefore, it is impossible for ℓ0 andℓ-1to intersect. This is contradicting the criterion that v0vℓ – 1 is an edge of G. So G contains no chordless 4 –cycle

Now we show that the complement of G satisfies the transitive orientation property. Let $\{I_v\}_{v \in V}$be an interval representation of G =(V,E). Define an orientation F of the complements for $\bar{G}$= (V,$\bar{E}$) as (xy) ϵ F ⟺ I_x<I_y(∀xy∈$\bar{E}$). Here I_x<$I_y$means that the interval $I_x$lies entirely on the left of the interval I_y . When I_x<I_y<I_z implies that I_x<I_z. This show that (x,z) ϵ F. That is (xy) ϵ F, (y,z) ϵ F ⟹(x,z) ϵ F. Thus, F is a transitive orientation of $\bar{G}$. Therefore$\bar{G}$ is a comparability graph.

(ii) ⟹ (iii)

Let us assume that G = (V,E) contains no chordless 4 – cycle, and let F be a transitive orientation of the complement $\bar{G}$.

Lemma A **:**

Let A_1 and A_2bemaximal cliques of G

(a) There exist an edge in F with one endpoint in A_1 and the other endpoint in A_2 .

(b) All such edges of $\bar{E}$ connecting A_1 with A_2 have the same orientation in F.

Proof of Lemma A

(a) If no such edge exists in F, then $A_1 \cup A_2$ is a clique of G, contradicting maximality.Suppose (ab) ϵ F and (dc) ϵ F with a, c ϵ A_1 and b, d ϵ A_2. We must show a contradiction. If either a = c or b = d, then transitivity of F immediately gives a contradiction; otherwise, these four vertices are distinct and (ad) or (bc) is in $\bar{E}$, since E may not have a chordless 4 – cycle. Without lose of generality, we assume that (ad) ϵ $\bar{E}$.We want to find which way it is oriented. Using the transitivity of F, ad ϵF(respectively (da) ϵ F) would imply ac ϵ F (respectively (db) ϵ F). Which is impossible, and lemma is proved.

(b) Consider the following relation on the collection $\mathcal{C}$ of maximal cliques: A_1< A_2iff there is an edge of F connecting A_1 with A_2 which is oriented towards A_2. By lemma A, this defines a tournament on $\mathcal{C}$. We claim that($\mathcal{C}$, <) is a transitive tournament, and hence linearly order $\mathcal{C}$. For suppose A_1< A_2 and A_2< A_3. Then there would be edges (wx) ϵ F and (yz) ϵ F with w ϵ A_1, x,yϵ A_2 and z ϵ A_3. If either (xz) ∉ E or (wy) ∉ E, then (wz)ϵ F and A_1< A_3. Therefore, assume that the edges (wy), (yx) and (xz) are all in E .Since G contains no chordless 4 – cycle, wz ∉ E, and the transitivity of F implies (wz) ∈ F. Thus A_1< A_3. This proves the transitive tournament claim.

Assume that $\mathcal{C}$ has been linearly ordered A_1,A_2,…….A_m according to the above relation. Suppose there exist cliques A_i<A_j<A_k with x ∈ A_i, x ∉ A_j, and x ∈ A_k. Since x ∉ A_j, there is a vertex y ∈ A_j such that (xy) ∉ E. But A_i<$A_j$implies (xy) ∈ F, where as A_j<A_k implies (yx) ∈ F, contradiction. This proves (iii).

(iii) ⇒ (i)

For each vertex x ∈ V, let $I_{(x)}$ denote the set of all maximal caliques of G which contain x. The sets $I_{(x)}$, for x ∈ V, are intervals of the linearly ordered set ($\mathcal{C}$, <). Now we have to show that (xy) ∈ E ⟺ $I_{(x)} \cap I_{(y)} \neq \emptyset$ (x,y∈ V).This holds, since two vertices are connected if and only if they are both contained in some maximal clique.

Corollary

An undirected graph G is an interval graph if and only if G is a triangulated graph and its complement $\bar{G}$ is a comparability graph.

Statement (iii) of the above theorem has an interesting matrix formulation.

Matrix whose entries are zeros and ones,is said to have the consecutive1's property for columns if itsrows can be permuted in such a way that the 1's in each column occur consecutively. In figure 3 the matrix M1 has the consecutive l's property for columns since its rows can be permuted to obtain M2. Matrix M3 does not possess this property.

$$\underset{M_1}{\begin{bmatrix} 1\,0\,0\,1\,1\,1 \\ 0\,1\,1\,1\,1\,1 \\ 1\,1\,1\,0\,0\,1 \\ 10\,1\,0\,0\,1 \\ 0\,1\,0\,1\,1\,1 \\ 0\,1\,1\,0\,1\,0 \end{bmatrix}} \longrightarrow \underset{M_2}{\begin{bmatrix} 1\,0\,1\,0\,0\,1 \\ 1\,0\,0\,1\,1\,1 \\ 0\,1\,0\,1\,1\,1 \\ 0\,1\,1\,1\,1\,1 \\ 0\,1\,1\,0\,1\,0 \\ 1\,1\,1\,0\,0\,1 \end{bmatrix}} \underset{M_3}{\begin{bmatrix} 0\,1\,0\,1\,0\,1 \\ 1\,1\,0\,0\,1\,0 \\ 1\,1\,0\,1\,1\,0 \\ 1\,1\,1\,1\,1\,0 \\ 0\,0\,1\,1\,1\,0 \\ 0\,0\,0\,1\,1\,0 \end{bmatrix}}$$

Figure 3

Theorem 2(Fulkerson) [6]

An undirected graph G is an interval graph if and only if its clique matrix M (maximal cliques – verses – vertices incidence matrix) has the consecutive 1's property for columns.

Proof:

An ordering of the maximal cliques of G corresponds to a permutation of the rows of M. Then the result follows from theorem1.

Asterodial Triple:[25] Three non-adjacent vertices are called an asterodial triples if they can't be ordered in such a way that every path from the first vertex to the third vertex passes through the neighbor of the second vertex. Figure 4 is an example of Asterodial Triples.

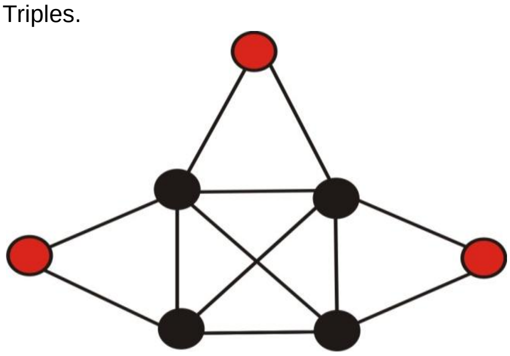

Figure 4

Another characterization of interval graph is given as follows

Theorem 3:(Lekkerkerker and Boland .) [7]

An undirected graph G is an interval graph if and only if the following two conditions are satisfied

(i) G is an triangulated graph

(ii) Any three vertices of G can be ordered in such a way the every path from the first vertex to the third vertex passes through a neighbor of the second vertex.Condition (ii) illustrates a well-known law of the business world. Every shipment from a supplier to the consumer must pass by the middleman.

The Graph is an interval graph ifand only if it is a Chordal Graph containing no asterodial triples.[25]

The Complexity of Consecutive L's Testing[10], [11].

Interval graphs were characterized as those graph whose clique matrices satisfy the consecutive l's property for columns. We may apply this characterization to a recognition algorithm for interval graphs G = (V,E) in a two step process. First verify that G is triangulated and, if so, enumerate its maximal cliques. This can be executed in time proportional to $|V|$ + $|E|$ and will produce at most n = $|V|$ maximal cliques Second, test whether or not the clique can be ordered so that those which contain vertex v occur consecutively for every v ∈V. That is, to list for the consecutive l's property for columns of the clique matrix, M – (0,1) valued with m rows and n columns and f zeros can be tested for consecutive l's property in O(m + n +f) steps.[1],[2]. Thus, this step can also be executed in linear time.For further study of complexity analysis in recognition of interval graphs are given in [21], [22].

Application of Interval Graphs[10], [15].

Application 1.

Suppose $C_1,C_2,\ldots.C_n$ are chemical compounds which must be refrigerated under closely monitored conditions. If compound C_1 must be kept at a constant temperature between t_i and t'_l degrees, how many refrigerators will be needed to store all the compounds?

Let G be the interval graph with vertices $C_1,C_2,\ldots.C_n$ and connect two vertices whenever the temperature intervals of their corresponding compounds intersect. By the Helly property, If $\{C_{i_1}, C_{i_2}, \ldots. C_{i_k}\}$ is a clique of G, then the intervals {[t_{ij}, t'_{ij}] /j = 1,2….k} will have a common point of intersection, say t. A refrigerator set at a temperature of t will be suitable for storing all of them. Thus, a solution to the minimization problem will be obtained by finding a minimum clique cover of G.

Hellyproperty: A family $\{T_i\}$ $i\epsilon$ I of subset of a set T is said to satisfy the Helly property if J ⊆ I and $T_i \cap T_j \neq \emptyset$ for all i, j ϵ J implies that $\cap_J T_j \neq \emptyset$.

Application 2. [4]

Let X represent a set of distinct data items (records) and let $\mathcal{I}$be a collection of subset of X called inquiries. Can X be placed in linear sequential storage in such a way that the members of each I $\epsilon\mathcal{I}$are stored in consecutive locations ?when the storage layout is possible, then records pertinent to any inquiry can be accessed with two parameters, a starting pointer and a length. calls this the consecutive retrievalproperty : it is clearly a restatement of the consecutive arrangement property.

Conclusion

This paper studied the Interval graph as a special class of intersection graph and perfect graph .It gives the property of interval graphs by Gilmore 's theorem statement and proof.An undirected graph is an Interval graph iff its clique matrix has the consecutive1's property for column.The complexity for the consecutive 1s testing can be executed in linear time. For an interval graph any three vertices can be ordered insuch a way that every path from the first vertex to the third vertex passes through a neighbor of the second vertex . Interval graph is a chordal graph without any asterodial triples. Interval graphs can be extensively used for the study of mutations of DNA in molecular biology, scheduling and communication network.

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